Mathematical Theory Behind Bioelectromagnetism

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Abstract: Nerve and muscle tissue are electrically active and act as generators of electricity. Body tissues are conductors of electricity. Hence the bioelectric sources give rise to currents throughout the body and on the body surface. Calculation of these currents then depends on the solution of a volume conductor problem. The currents also give rise to magnetic fields. Several aspects of this overall problem are of interest. 1) What is the relation of potentials in and on the surface of the volume conductor to the bioelectric sources? 2) What is the relation of external magnetic fields to these sources? 3) How can these sources be related to the electrophysiology of the cell? 4) Given measurements of the body surface potentials or extracorporeal magnetic fields, how can one deduce the sources which give rise to the fields (inverse problem)? This paper will attempt to sketch the mathematical theory of this problem.

VOLUME CONDUCTOR PROBLEM

Plonsey has provided an excellent treatment of bioelectricity in his classic book [1]. This book was followed by a revised version in textbook form [2]. Last year **Bioelectromagnetism** by Malmivuo and Plonsey was published [3]. The present paper follows most closely my tutorial paper [4]. References to primary sources can be found in these publications, and generally will not be provided here.

A model of the electrocardiogram must provide a characterization of the bioelectric sources and the volume conductor in which they are immersed. Body tissues are conductors of electricity. Hence the electromotive forces which arise in the heart muscle (myocardium) as a result of electrical activity of cardiac cells give rise to electric fields and currents throughout the torso, and therefore to electric potentials on the skin. The relationships between the sources and the currents and fields which they produce are governed by well established laws of electricity, and constitute the volume conductor problem. To complete the model the sources, or electromotive forces, throughout the heart must be related to electrical activity at the cellular level.

In the frequency range to 1000 Hz the average thoracic impedance is to an excellent approximation resistive with a conductivity of about 0.2 S/m. Individual tissues have conductivities which are all within an order of magnitude of this value. The magnetic permeability of the tissues is very close to that of free space, μ_o. Furthermore, for the current densities which exist as a result of bioelectric activity, the conductivity is constant. Hence the volume conductor is linear and resistive. Linearity implies that the potential arising from several sources is equal to the sum of the potentials contributed by each of the sources acting alone.

Another important feature of the volume conductor problem is that it is quasi-static which means that currents and potentials throughout the body are instantaneously related to sources in the heart. Since the electromagnetic problem is quasi-static the electric and magnetic fields are decoupled. The electric field is proportional to the gradient of the electric scalar potential. The divergence of the current density is zero. The currents and potentials in the body at each instant of time depend on the source distribution at that instant and not on any previous history. The magnetic field is related instantaneously to the currents in the volume conductor through the Biot-Savart law.

The above conditions are summarized in mathematical form as follows. Let **J** be the current density and V be the electric scalar potential (voltage) at a point in the volume conductor. Then

$$\boldsymbol{J} = -\sigma \nabla V \qquad (1)$$

where σ is the conductivity of the tissue at the point. In the absence of an external source of current,

$$\nabla \cdot \boldsymbol{J} = 0 \qquad (2)$$

Bioelectric sources are associated with the movement of ions across the cell membrane, and involve the conversion of chemical energy to electric energy. The sources can be represented as an "impressed" current source density, $\mathbf{J}^i$.

$$\boldsymbol{J} = -\sigma \nabla V + \boldsymbol{J}^i \qquad (3)$$

where $\mathbf{J}^i$ vanishes outside the heart. The impressed current density is equivalent to a source current dipole moment per unit volume [4].

The torso as a volume conductor is linear, inhomogeneous, resistive, anisotropic, and bounded. It is reasonable to consider it to be divided into discrete regions such as heart, lung, bone, blood, muscle, etc. with an appropriate conductivity assigned to each region. The heart contains the

sources of electricity, or electromotive forces, $\mathbf{J}^i$.

Strictly as a consequence of linearity, the potential, V, at an arbitrary point on the surface of the body (or anywhere in the volume conductor) is related to the distribution of current sources, $\mathbf{J}^i$, throughout the heart as follows.

$$V = \int \nabla Z \cdot \boldsymbol{J}^i dv \approx \sum \nabla Z_i \cdot (\int \boldsymbol{J}^i dv_i) = \sum \nabla Z_i \cdot \boldsymbol{p}_i \qquad (4)$$

This equation is a statement of superposition. ∇Z is a transfer impedance which relates the source $\mathbf{J}^i$ in the element of heart volume, dv, to the potential V at the observation point. Z will depend on the location of the source, the location of the observation point, the reference point, and the shape and conductivity of the torso and its internal structures. The fact that it is the gradient of a scalar follows from considerations of reciprocity, and will be demonstrated in the next section.

It is sometimes convenient to divide the heart into a number of regions, and to assign a lumped dipole, $\mathbf{p}_i$, to the centroid of each region. This lumped dipole would be found from the integration of $\mathbf{J}^i$ over the ith region. See Eq. (4). In this case the cardiac electric sources are represented by a finite number of lumped current dipoles, i.e. a multiple dipole source.

GREEN'S THEOREM

Since the volume conductor problem is quasi-static, charges are instantaneously redistributed at the interfaces between regions. The potential and the normal current density are continuous at surfaces separating regions of different conductivity. Since the body is effectively insulated, the normal current density at the outer surface, the skin, is zero. Since at each instant the problem is static, it is useful to employ Green's theorem. For simplicity we will write the equations for the case of a homogeneous volume conductor.

$$\sum \int [\sigma'(\psi' \nabla \phi' - \phi' \nabla \psi') - \sigma''(\psi'' \nabla \phi'' - \phi'' \nabla \psi'')] \cdot dS = \sum \int [\psi \nabla \cdot \sigma \nabla \phi - \phi \nabla \cdot \sigma \nabla \psi] dv \qquad (5)$$

where dv is an element of volume, dS is an element of area on a conductivity interface, and ϕ and ψ are two scalar functions which are well behaved in each region. We adopt the convention that dS is directed from the primed region to the double primed one.

Let ϕ be the electrocardiographic potential. Various results of interest can be obtained by appropriately selecting ψ. Initially we choose ψ to be the potential associated with a unit current source at point Q and a unit current sink at Q' in the volume. Both ϕ and ψ satisfy the boundary conditions at the interfaces and skin. Hence the term on the left of Eq. (5) vanishes, and

$$\phi(Q) - \phi(Q') = -\int \psi \nabla \cdot \boldsymbol{J}^i dv = \int \boldsymbol{J}^i \cdot \nabla \psi dv \qquad (6)$$

where the integral on the right follows from vector analysis.

Suppose Q and Q' are two points on the body surface which we will designate 1 and 2 respectively. Then the potential difference V_{12} in this lead is

$$V_{12} \equiv \phi_1 - \phi_2 = \int \boldsymbol{J}^i \cdot \nabla \psi dv \qquad (7)$$

A comparison of this equation with Eq. (4) indicates that $\nabla \psi$ is identical to the transfer impedance, ∇Z, and has the dimensions ohm/length. This vector function is the negative of the electric field in the volume conductor which results when unit current is injected into the positive terminal of the lead and removed from the negative terminal, and is termed the lead field [5]. The above argument then constitutes a proof of the reciprocity theorem initially proved by Helmholtz [6]. It also demonstrates that the transfer impedance is the gradient of a scalar.

Let us now consider the case where $\psi = 1/r$ where r is the distance from an observation point to dv or dS. Note that $\nabla(1/r) \cdot dS = d\Omega$ where $d\Omega$ is the solid angle subtended by dS at the observation point. Eq. (5) then becomes

$$4\pi\sigma\phi = \int \boldsymbol{J}^i \cdot \nabla(1/r) dv - \int (\sigma' - \sigma'') \phi d\Omega \qquad (8)$$

In the term on the left σ and ϕ are the values of conductivity and potential at the observation point. The first term on the right is $4\pi\sigma$ times the potential that would exist if the cardiac (primary) sources were in an unbounded homogeneous medium. The integral in the second term on the right is taken over the outer surface together with all internal surfaces separating the discrete regions of the volume conductor. This term accounts for the secondary sources. Eq. (8) provides an implicit equation for determining the solution to the volume conductor problem numerically, i.e. the boundary element method.

Another result developed by Helmholtz in a sense subsumes Thevenin's theorem, which it antedated. Consider Eq.(8) for the special case of a bounded homogeneous volume conductor of conductivity, σ Let the observation point lie outside the volume where σ is zero, and let V° be the potential on the insulated boundary.

$$\int \boldsymbol{J}^i \cdot \nabla(1/r) dv - \int \sigma V^\circ d\Omega = 0 \qquad (9)$$

The first term is just $4\pi\sigma$ times the potential in an unbounded homogeneous conductor. Hence the potential in an unbounded medium is just the potential arising from a double layer on the boundary where the moment of the double layer is proportional to the potential appearing on the insulated boundary.

BIDOMAIN MODEL

The bidomain model provides the basis for relating $\mathbf{J}^i$ to the cellular action potential [7]. Following is a summary of the theory. The current densities, **J**, within the intracellular (I) and extra-cellular (e) domains are given by

$$J_i = -\sigma_i \nabla \phi_i \qquad (10)$$

$$J_e = -\sigma_e \nabla \phi_e \qquad (11)$$

where σ is the conductivity of the domain and ϕ is the potential. Current leaving one domain must enter the other by crossing the membrane. Hence the membrane current per unit volume, I_m is given by

$$I_m = -\nabla \cdot J_i = \nabla \cdot J_e = -\nabla \cdot (\sigma_e \nabla \phi_e) \qquad (12)$$

Since the bioelectric problem is quasi-static,

$$\nabla \cdot J = \nabla \cdot (J_i + J_e) = 0 \qquad (13)$$

The transmembrane potential, $\phi_m = \phi_i - \phi_e$. Therefore

$$-\nabla \cdot (\sigma_i \nabla \phi_m) = \nabla \cdot (\sigma_H \nabla \phi_e) \equiv \nabla \cdot J^i \qquad (14)$$

where σ_H is taken to be the sum of the intracellular and extracellular conductivities, and we define a current source density, $\mathbf{J}^i$

$$J^i = -\sigma_i \nabla \phi_m \qquad (15)$$

The impressed current sources within the electrically active myocardium are thus proportional to the spatial gradient of the transmembrane potential. Note that Eq. (14) can be generalized to the entire volume conductor.

$$\nabla \cdot (\sigma \nabla V) = \nabla \cdot J^i \qquad (16)$$

where σ is the tissue conductivity and we have designated the extracellular potential at any point as V. $\mathbf{J}^i$ is zero outside the heart.

MAGNETIC FIELD

A treatment of the theory of the magnetocardiogram was given by Baule and McFee from the standpoint of lead field theory [8]. The idea is that the coil used to detect the magnetic field is reciprocally energized with unit current, creating a lead field in the heart. The following analysis presents an expression for the magnetic field outside the body, taken to be an inhomogeneous conductor, in terms of internal sources and the potentials at surfaces of discontinuity [9].

With the aid of Eq. (3) we can write

$$H = \frac{1}{4\pi}\left[\int J^i x \nabla\left(\frac{1}{r}\right) dv + \sum \int \sigma_i \nabla V x \nabla\left(\frac{1}{r}\right) dv\right] \qquad (17)$$

where the summation on the right is over all regions of the body. The term on the right can be transformed with the use of vector analysis [9] to give

$$H = \frac{1}{4\pi}\int J^i x \nabla\left(\frac{1}{r}\right) dv + \frac{1}{4\pi}\sum \int (\sigma' - \sigma'') \nabla\left(\frac{1}{r}\right) x dS_j \qquad (18)$$

The term on the right involves integrals over all surfaces $\mathbf{S}_j$ separating regions of different conductivity, σ' and σ", where the vector $d\mathbf{S}_j$ is taken to be directed from the primed region to the double primed one. The results depend on the fact that both the potential V and its derivative normal to the surface of discontinuity, i.e. perpendicular to $d\mathbf{S}_j$ are continuous. At the external boundary σ" is zero.

The first term on the right of Eq. (18) is referred to as the primary source while the second term is referred to as the secondary source. Much research and analysis has been devoted to the question of the relative magnitude of these two terms. Clearly if the secondary source contribution is small, then the magnetic field would reflect the sources in active tissue independent of the effects of the volume conductor. We will not discuss this topic here.

Another question of great interest is what new information is present in the magnetic field beyond that in the body surface electric potential distribution. It would appear that since the magnetic field is related to the curl of $\mathbf{J}^i$ while the electric field is related to its divergence, the electric and magnetic fields contain independent information. This simplistic conclusion is clouded somewhat by the fact that $\mathbf{J}^i$ is the gradient of a scalar. The conclusion may depend on whether we are considering the magnetocardiogram (MCG) or the magnetoencephalogram (MEG). We have shown that if anisotropy of heart muscle can be neglected, then ideally the MCG can be deduced from the ECG [10]. We will not discuss the details of this argument here.

MULTIPOLES

The potential in a lead is related to the sources through the transfer impedance or lead field, ∇Z. See Eq. (7). From reciprocity ∇Z is equal to the negative of the electric field resulting when unit current is injected into the positive terminal of the lead and removed from the negative terminal, and thus obeys the relation

$$\nabla \cdot \sigma \nabla Z = 0 \qquad (19)$$

It follows that ∇Z has zero divergence everywhere in the volume conductor except at surfaces separating regions of different conductivity, or in regions where the conductivity is anisotropic. At this point we will ignore anisotropy.

A particular choice of Z satisfying the condition that $\nabla^2 Z$ is zero is

$$Z_{nm} = (2-\delta_m^o)\frac{(n-m)!}{(n+m)!} r^n P_n^m(\cos\theta) e^{im\phi} \qquad (20)$$

where r,θ,ϕ are spherical coordinates of a point relative to the origin, δ_n^m is the Kronecker delta, and ${}_n^m P$ is an associated Legendre polynomial.

The scalar function, Z, for an arbitrary lead can be constructed from an infinite series of Z_{nm}. Define

$$a_{nm} + ib_{nm} = \int \mathbf{J}^i \cdot \nabla Z_{nm} dv \qquad (21)$$

It then follows that

$$V = \sum\sum(\alpha_{nm} a_{nm} + \beta_{nm} b_{nm}) \qquad (22)$$

Eq. (22) is the multipole expansion of the potential, V. The terms a_{nm} and b_{nm} are the coefficients of the multipole expansion of the source. Eq. (21) indicates how the multipoles are related to the distribution of current dipole moment in the heart. Note that b_{nm} is zero.

The inverse problem is that of deducing the source distribution, $\mathbf{J}^i$, from the surface potentials or external magnetic field. In general the inverse solution is not unique. Techniques for solving the inverse problem will not be discussed here. It might be noted that the multipole coefficients, a_{nm} and b_{nm} constitute a canonical compact description of the sources as well as the body surface potential distribution. In the case of a homogeneous volume conductor, the lead with lead field ∇Z in the heart region must be the lead with this lead field everywhere in the volume conductor. It follows that

$$a_{nm} + ib_{nm} = \int \sigma V \nabla Z_{nm} \cdot dS_o \qquad (23)$$

where S_o is the surface of the body. In principle, this equation provides an alternative approach to determining the multipolar coefficients by integrating the potential over the body surface. manner. The heart vector is the first order term in the multipole expansion.

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